JOURNAL OF COMPUTATIONAL PHYSICS 3, 148–153 (1968)

NOTE

Numerical Computation of Electric and Magnetic Fields in a Uniform Waveguide of Arbitrary Cross Section

The author has recently written a computer program that numerically solves for the electric and magnetic fields in a uniform waveguide of arbitrary, simply connected cross section.[1] While this program accomplishes the same objective as that of a computer program by Davies and Muilwyk [1], the algorithm is quite different.

Either the axial electric field of a TM mode or the axial magnetic field of a TE mode can be represented by ϕ in the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -k_c{}^2 \phi \tag{1}$$

when appropriate boundary conditions are applied ([2], p. 316). For a TM mode, the electric field equals zero at the waveguide boundary; for a TE mode, the derivative of the axial magnetic field normal to the waveguide boundary equals zero. An approximation to ϕ, ψ, is computed numerically at a number of interior points arranged in a rectangular pattern within the walls of the waveguide. The points in the rectangular pattern that are nearest the waveguide wall are taken to be boundary points. The partial derivatives in (1) are replaced by finite-difference approximations ([3], p. 162). Then for each interior point, P, not adjacent to a boundary point, we have

$$4\psi_P - \psi_A - \psi_B - \psi_L - \psi_R = \tau\psi_P, \tag{2}$$

where ψ_P is taken at P; ψ_A, ψ_B, ψ_L, and ψ_R are taken at interior points above, below, to the left, and to the right of P. Equations that apply to interior points adjacent to boundary points differ somewhat from (2) in order to approximate the boundary conditions mentioned above. In (2)

$$\tau = (k_c h)^2 = \left(\frac{2\pi h}{\lambda_c}\right)^2, \tag{3}$$

[1] This program is written in ALGOL for the Burroughs B-5500 computer. The author will be glad to mail either a program listing, or a more extensive description of the algorithm upon request.

where h is the spacing between adjacent interior points and λ_c is the cutoff wavelength. This system of equations is linear in ψ and can therefore be expressed in matrix vector from as

$$Av = \tau v, \tag{4}$$

where A is a square matrix defined by (2) and the boundary conditions, and v is a vector whose elements are the values of ψ at all the interior points.

From (4) we see that we have an algebraic eigenproblem to solve, and that each eigenvalue of A, and its eigenvector correspond to a waveguide mode. Furthermore, the eigenvalue τ can be used with (3) to compute the cutoff wavelength, and the eigenvector v defines the axial electric or magnetic field of the mode at all interior points. Because the TE and TM modes have different axial field boundary conditions, their matrices differ, even for the same waveguide cross section, step spacing, and set of boundary points. Because the modes of greatest interest in engineering and scientific applications are the dominant TE and TM modes, the program computes only these modes, which correspond to the lowest eigenvalues of the appropriate matrices and their eigenvectors. (The program can be modified for computation of higher waveguide modes as well.)

The program works only for positive-definite matrices. While the matrices used for the TM modes are positive-definite (see below), a matrix which would be used to compute a TE mode axial magnetic field over a full waveguide cross section is singular. For this reason the program only computes TE modes for the waveguides having a plane symmetry along which the axial magnetic field of the dominant TE mode is zero.

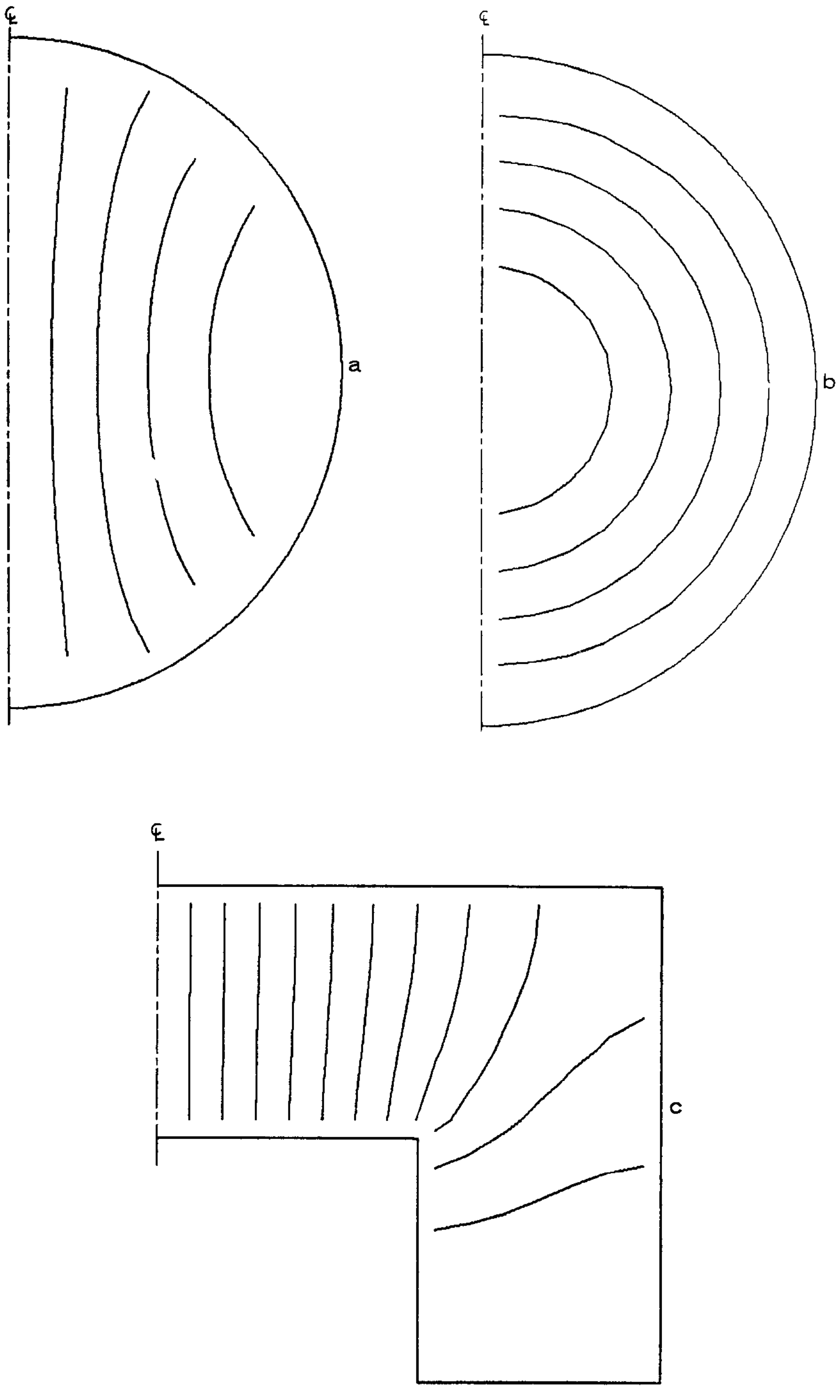
In all, the program uses matrices of three types: (a) for the TM-mode axial electric field over the entire waveguide cross section; (b) for the TM-mode axial electric field over one-half the cross section of a symmetric waveguide; and (c) for a TE-mode axial magnetic field over one-half the cross section of a symmetric waveguide. Matrices of all three types are symmetric, sparse, irreducibly diagonally dominant, and positive-definite ([3], p. 23, Definition 1.7), and their lowest eigenvalues are simple.[2]

ALGORITHM

The eigenvector of the lowest eigenvalue is computed by an algorithm discussed by Wilkinson ([4], p. 142). A sequence of vectors, $x^{(i)}$, is computed by

$$(A - \tau_x I)x^{(i+1)} = x^{(i)} \qquad i = 1, 2, 3..., \qquad x^{(1)} = (1, 1, 1,..., 1), \tag{5}$$

[2] The author can provide proofs of these matrix properties.



℄
a
℄
b
℄
c

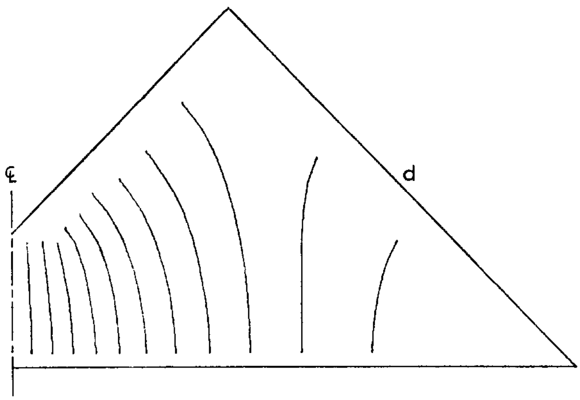


FIG. 1.

where τ_x is a real constant for which

$$| \tau_1 - \tau_x | < | \tau_j - \tau_x | \qquad j > 1 \tag{6}$$

and

$$\tau_1 < \tau_2 \cdots < \tau_n$$

are the eigenvalues of A. Then the sequence $x^{(i)}$ converges to an eigenvector corresponding to τ_1, and since τ_1 is simple, this eigenvector is uniquely determined by A. To compute τ_1, we derive from (5) that

$$\tau_1 = \tau_x + \lim_{i\to\infty} \frac{\| x^{(i)} \|_\infty}{\| x^{(i+1)} \|_\infty} \tag{7}$$

The algorithm consists of the following steps in sequence.

(1) The boundary points are computed automatically on the basis of the waveguide cross section and step spacing read in.

(2) The constant τ_x is determined to be less than τ_1, but close enough to it to produce a rapid convergence of the eigenproblem algorithm described above. For TM modes, τ_x is computed automatically; for TE modes it is read in.[3]

(3) The matrix $(A - \tau_x I)$ and then the lower triangular matrix L are computed, where

$$A - \tau_x I = LL^{\mathrm{T}}. \tag{8}$$

[3] If τ_x as read in is too large, so that $\tau_x \geqslant \tau^1$, then $(A - \tau_x I)$ is not positive-definite, and the matrix L in (8) cannot be computed. In this case, the computer automatically reduces τ_x to 0.9 of its previous value, prints a message to this effect, and repeats the computation of L.

For (8) we use the facts that the matrix $(A - \tau_x I)$ is symmetric (since A is symmetric) and positive-definite (since $\tau_x < \tau_1$).

Only the elements of L (which is sparse) not known to be zero are stored in computer memory. (L^{T} need not be stored.)

(4) Equation (5) is solved repeatedly using the triangular matrix L. That is, we solve for $i = 1, 2, 3\ldots,$

$$Ly^{(i)} = x^{(i)}, \tag{9}$$

$$L^{\mathrm{T}} x^{(i+1)} = y^{(i)}, \tag{10}$$

which are derived from (5) and (8). This process is terminated when

$$\| a x^{(i+1)} - x^{(i)} \|_\infty$$

is less than some predetermined amount, where

$$\| x^{(i)} \|_\infty = 1$$

and a is a scalar for which

$$\| a x^{(i+1)} \|_\infty = 1.$$

The last vector of the sequence is used as the eigenvector; the last two vectors are used with (7) to compute the eigenvalue τ_1.

(5) Using the computed τ_1, the cutoff wavelength is computed from (3).

(6) The program provides for automatic plotting of the waveguide outlines and transverse field lines.

Program Output

Figure 1 shows the waveguide field plots obtained in four of the test cases. Fig. 1a and 1b give the dominant TE and TM modes, respectively, in circular waveguides; Figures 1c and 1d give the dominant TE modes in two ridged waveguides. Transverse magnetic field lines are shown in the TM-mode plot; transverse electric field lines are shown in the TE-mode plots. In these test runs, between 100 and 150 interior points were used. In the tests on the dominant TE and TM modes in rectangular and circular waveguides, the maximum difference between the computed cutoff wavelengths and those obtained from the analytic formulas is 0.3 %.

Acknowledgment

The author appreciates the advice and support of Professor John G. Herriot, Computer Science Department, Stanford University, toward the development of the computer program.

References

1. J. B. Davies and C. A. Muilwÿk, Numerical solution of uniform hollow waveguides with boundaries of arbitrary shape, *Proc. IEE (London)* **113**, 277 (1966).
2. S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio" (2nd ed.). Wiley, New York (1953).
3. R. S. Varga, "Matrix Iterative Analysis." Prentice-Hall, Englewood Cliffs, New Hersey (1962).
4. J. H. Wilkinson, "Rounding Errors in Algebraic Processes." Prentice-Hall, Englewood Cliffs, New Jersey (1963).

Charles W. Steele

Ampex Corporation, Redwood City, California